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Techniques for treating non-Gaussian random processes

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TECHNIQUES FOR TREATING

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I. INTRODUCTION

The primary objective of this paper is the solution of the following problem:

If a non-gaussian continuous random process is the input to an arbitrary linear filter, what is the probability density function of the filter output?

The technique used permits an approximate solution which can be made arbitrarily close to an exact solution at the expense of computation time and effort. The approximate solution is accomplished by forming a rectangular-pulse model of the continuous process and then treating that model.

In some cases, the methods used allow for the solution to the inverse of the above problem. That is, if the output distribution is known, and the spectrum of the input is very wide with respect to the filter range, the input distribution can be determined.

The form of this model, when it is a Markov process, also lends itself to the solution of the axis crossing distributions associated with the process.

The most significant work published in this area has been accomplished by Darling and Siegert (3). They have developed an integral solution for a class of situations which includes the above problem if the input to the filter is a Markov process. Their derivation was aimed primarily at determining the probability density function for the output of a first-order filter, a square-law detector, and an arbitrary linear filter in series given a white-noise input. This problem was then generalized to include other than square-law functions. Thus, for the case where the square-law detector is replaced by a linear device, their solution is equivalent to determining the distribution of the output of an arbitrary filter, given a continuous Markov process as an input. These integral equations are at best, however, difficult to solve and the technique in general is limited to Markov inputs only.

McFadden (10) has used these and other techniques to determine the distribution of the output of an R-C filter if the input is a binary random process. Wohnam (17) has also developed a technique to determine the probability density of the output of a low-pass system when the input is a Markov step process with a finite number of states.

The axis-crossing problem has received more attention. Rice, (13) in a comprehensive mathematical treatment of Gaussian noise, developed an expression for p(t) dt, the probability that a Gaussian random variable goes through zero (though not necessarily for the first time) in t, t + dt, when it is known that the variable goes through zero at t = 0 with a slope opposite to that at t. For narrow-band noise p(t) will be very close to the distribution of first-crossing durations for sufficiently small t. In general, Rice's treatment of the axis-crossing problem requires knowledge of the joint distribution of the process and its derivative which, for many processes, does not even exist.

Siegert (14) has solved for the first-passage probability density $P(y_o | t,a)dt$, the probability that a variable y(t) passes the value a for the first time in the interval t, t + dt if $y(0) = y_o$, on the condition that y(t) is a Markov process. McFadden (8) has also determined expressions for the axis-crossing intervals for a stationery Gaussian Process, and several authors have investigated the first-passage problem for discrete processes as a form of the Gamblers-Ruin Problem.

With the exceptions of Darling and Siegert's work on the filter output distribution and Siegert's work on the first-passage problem, most of the work in these two areas has been concentrated on the Uhlenbeck process and the Wiener-Einstein Process (16) both of which are normally distributed, or on discrete-valued processes. Hopefully, the techniques developed in this paper, which are markedly different from Darling and Siegert's methods, will prove useful for obtaining approximate solutions for classes of problems not covered by the works outlined above.

II. MODEL FOR A FIRST-ORDER PROCESS

A. The Difference Equation.

Assume a random process generated by a series of adjacent rectangular pulses of equal duration t whose magnitudes are related by

$$x_n = z_n + \alpha x_{n-1} \tag{1}$$

where

x_n = magnitude of the nth pulse

x_{n-1} = magnitude of the (n-1) pulse

z_n = an independent random variable

α = a positive constant less than one.

From Equation 1, certainly $x_{n-1} = z_{n-1} + \alpha x_{n-2}$; therefore

$$\begin{aligned} x_n &= z_n + \alpha z_{n-1} + \alpha^2 x_{n-2} \\ &= z_n + \alpha z_{n-1} + \alpha^2 z_{n-2} + \alpha^3 z_{n-3} + \text{--------} \end{aligned} \tag{2}$$

where z_n, z_{n-1}, z_{n-2} ---- are specific values of the random variable z.

B. The Autocorrelation Function.

Using Equation 2, the autocorrelation function for this process can be determined. First, the ensemble average of the function multiplied by itself, i.e. $\phi(o)$, the autocorrelation function for $\tau = 0$ will be found.

$$\widetilde{x_n \cdot x_n} = \widetilde{(z_n + \alpha z_{n-1} + \alpha^2 z_{n-2} \text{ ---})(z_n + \alpha z_{n-1} + \alpha^2 z_{n-2} \text{ ---})}$$

Since $\widetilde{z_n \cdot z_{n-m}} = 0$ when m = 0 this averaging process results in

$$\widetilde{x_n \cdot x_n} = \widetilde{z_n \cdot z_n} + \alpha^2 \widetilde{z_{n-1} \cdot z_{n-1}} + \alpha^4 \widetilde{z_{n-2} \cdot z_{n-2}} \text{ ---}$$

$$= \widetilde{z^2} \sum_{n=0,1,2---} \alpha^{2n}$$

Now, denote the autocorrelation function evaluated at $t = \Delta t$ as $\phi(1)$; then

$$\phi(1) = \widetilde{x_n \cdot x_{n-1}}$$

$$= \widetilde{(z_n + \alpha\, z_{n-1} + \alpha^2\, z_{n-2} \text{ ---})(z_{n-1} + \alpha\, z_{n-2} + \alpha^2\, z_{n-3} \text{ ---})}$$

$$= \alpha\, \widetilde{z^2}\, (1 + \alpha^2 + \alpha^4 \text{ ---})$$

$$= \alpha(\widetilde{z^2} \sum_{n=0} \alpha^{2n})$$

Similarly,

$$\phi(2) = \alpha^2(\widetilde{z^2} \sum_{n=0} \alpha^{2n})$$

and, in general

$$\phi(j) = \alpha^j\, (\phi(o))$$

This defines the value of the autocorrelation function at integral numbers of pulse durations. Because the process is made up of rectangular pulses connecting these values with straight lines will complete the autocorrelation function description. Figure 1 illustrates this approximation.

If α is now defined to be $e^{-\theta\Delta t}$ where Δt is the pulse duration of the process,

$$\phi(j) = \left[e^{-\theta\Delta t}\right]^j \phi(o) = e^{-\theta j\Delta t}\, \phi(o)$$

Then if $\tau = j\Delta t$ and Δt is made arbitrarily small

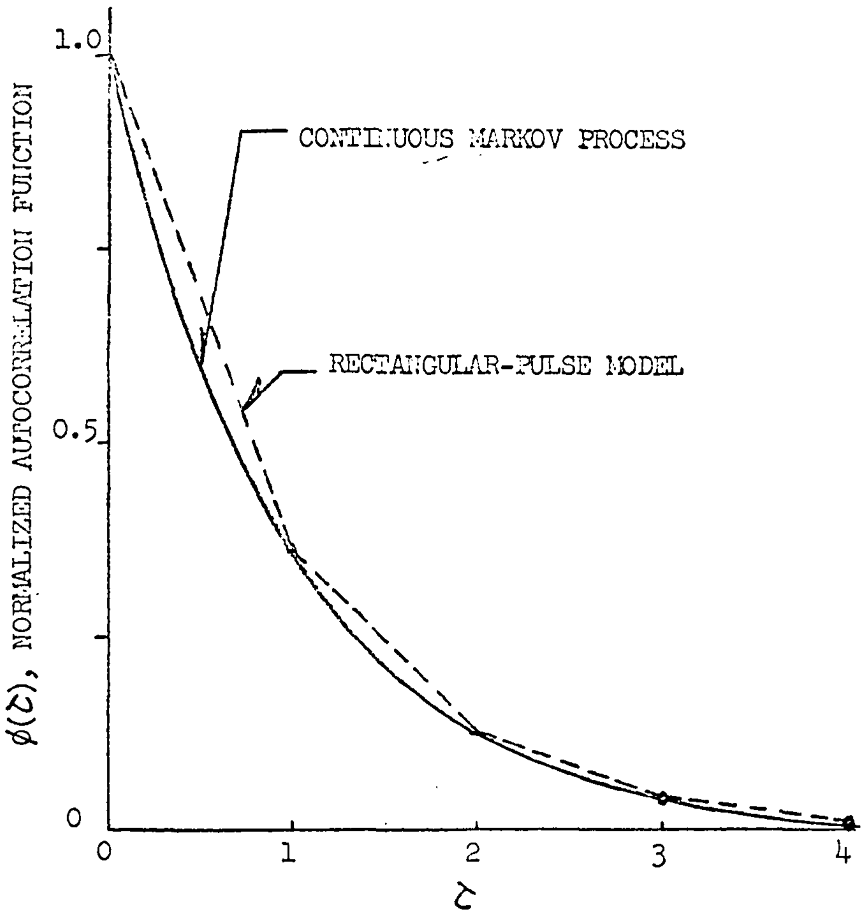


Figure 1. Comparison of the autocorrelation functions for a continuous Markov process and for the rectangular pulse model used as an approximation to the continuous process. This particular case represents the case calculated in Example 5 where $\alpha = 0.368$

$$\phi(\tau) = e^{-\theta\tau}\phi(o)$$

It seems reasonable therefore, that this model can be used to represent a continuous Markov process in the limiting case where Δt is small.

C. Conditional Probability Density Functions.

Equation 1 states that $x_n = z_n + \alpha\, x_{n-1}$, and if the process is assumed to be time stationery (when sampled at points separated by the pulse width) then the distribution of x_{n-1} can be considered the same as x_n. Therefore, defining

$p_o(x)$ = probability density function for x

$P_o(w)$ = Fourier transform of p_o.

$p_z(z)$ = probability density function for z.

$P_z(w)$ = Fourier transform of $p_z(z)$.

it can be shown (6) from Equation 1, that

$$P_o(w) = P_z(w)\ P_o(\alpha w) \qquad (4)$$

Thus, the characteristic function for the probability density function of the random variable z is defined as

$$P_z(w) = \frac{P_o(w)}{P_o(\alpha w)} \qquad (5)$$

This result can be used to specify the first conditional probability density function for x_n, given x_{n-j},

$$p\left\{x_n \mid x_{n-j}\right\} = p_1(x_n, j).$$

Since

$$x_n = z_n + \alpha x_{n-1}$$

then

$$p_1(x_{n,1}) = \alpha(x_{n-1}) + p_z(x_n) \qquad (6)$$

Further,

$$x_n = z_n + \alpha z_{n-1} + \alpha^2 x_{n-2}$$

then, if x_{n-2} is given,

$$x_n - \alpha^2 x_{n-2} = z_n + \alpha z_{n-1}$$

and therefore, the distribution of $(x_n - \alpha^2 x_{n-2})$ is $P\left\{x_n \mid x_{n-2}\right\}$ shifted by an amount $\alpha^2 x_{n-2}$ (See Figure 2). $P\left\{(x_n - \alpha^2 x_{n-2}) \mid x_{n-2}\right\}$ has characteristic function

$$P_{n|n-2}(w) = P_z(w) \cdot P_z(\alpha w)$$

and in general

$$P_{n|n-j}(w) = P_z(w)\, P_z(\alpha w)\, P_z(\alpha^2 w) \; --- \; P_z(\alpha^j w) \qquad (7)$$

$$= \sum_{n=0}^{j} P_z(\alpha^j w)$$

But since $P_z(w) = \dfrac{P_o(w)}{P_o(\alpha w)}$

$$P_{n|n-j}(w) = \left[\frac{P_o}{P(\alpha w)}\right] \cdot \left[\frac{P_o(\alpha w)}{P_o(\alpha^2 w)}\right] \cdot \left[\frac{P_o(\alpha^2 w)}{P_o(\alpha^3 w)}\right] -- \left[\frac{P_o(\alpha^j w)}{P_o(\alpha^{j+1} w)}\right]$$

$$= \frac{P_o(w)}{P_o(\alpha^{j+1} w)} \qquad (8)$$

This result states that the distribution of x_n given x_{n-j} has a distribution, shifted by an amount $\alpha^j x_{n-j}$, with characteristic function

$\dfrac{P_o(w)}{P_o(\alpha^{j+1}w)}$ where P_o is again the primary distribution of the process.

To determine the probability density function for the process at a time t given the value of the function at time 0, let there be j pulse durations in the duration t. Then

$$P_{x_t}(w)_{|x_o} = \frac{P_o(w)}{P_o(\alpha^{j+1}w)}$$

and, if $\alpha = e^{-\theta\Delta t}$, and $\Delta t = \frac{t}{j}$, substitution of these quantities gives

$$P_{x_t}(w)_{|x_o} = \frac{P_o(w)}{P_o\left[(e^{-\theta\Delta t})^{\frac{t}{\Delta t}+1}w\right]} = \frac{P_o(w)}{P_o\ e^{-\theta(t+\Delta t)}w}$$

As Δt is made arbitrarily small

$$P_{(x_t|x_o)}(w) = \frac{P_o(w)}{P_o(we^{-\theta t})} \tag{9}$$

and the conditional probability density function shifted by an amount $e^{-\theta t}x_o$ is the inverse transform

$$P\,(x_t - e^{-\theta t}x_o)\,x_o = \mathfrak{J}^{-1}\left[\frac{P_o(w)}{P_o(e^{-\theta t}w)}\right] \tag{10}$$

Some simple examples of first probabilities are calculated here and are shown in Figures 2 and 3.

Example 1.

Consider a process with probability density function

$$p_o = \beta e^{-\beta x} \qquad x \geq 0$$
$$= 0 \qquad x < 0$$

Then $P_o(w) = \frac{\beta}{\beta + jw}$, and using Equation 10:

$$\mathfrak{F}\left[P_{\left[x_t - x_o e^{-\theta t} \mid x_o\right]}\right] = \frac{\beta + jwe^{-\theta t}}{\beta + jw} = e^{-\theta t} + \frac{\beta(1-e^{-\theta t})}{\beta + jw}$$

Therefore

$$P_{\left\{(x_t - x_o e^{-\theta t}) \mid x_o\right\}} = e^{-\theta t}\,\delta(x_t) + \beta(1-e^{-\theta t})e^{-\beta x_t}$$

and

$$P\left\{x_t \mid x_o\right\} = e^{-\theta t}\,\delta(x_t - x_o e^{-\theta t}) + \beta(1-e^{-\theta t})e^{-\beta(x_t - x_o e^{-\theta t})}$$

These functions are shown in Figure 3.

Example 2.

Consider a process with a distribution

$$p_o = \beta/2\ e^{-\beta|x|}$$

Then

$$P_o(w) = \frac{\beta^2}{\beta^2 + w^2},$$

and from Equation 10:

$$\mathfrak{F}\left[P\left\{(x_t - x_o e^{-\theta t}) \mid x_o\right\}\right] = \frac{\beta^2 + w^2 e^{-2\theta t}}{\beta^2 + w^2}$$

$$= e^{-2\theta t} + \frac{\beta^2(1-e^{-2\theta t})}{(\beta^2 + w^2)}$$

which has an inverse

$$P\left\{(x_t - x_o e^{-\theta t}) \mid x_o\right\} = e^{-2\theta t}\delta(x_t) + \beta/2(1-e^{-\theta t})e^{-\beta|x_t|}$$

and therefore

$$P\left\{x_t \mid x_o\right\} = e^{-2\theta t}\delta(x_t - x_o e^{-\theta t}) + \beta/2(1-e^{-2\theta t})e^{-\beta|(x_t - x_o e^{-\theta t})|}$$

The primary and resulting conditional probability density functions for this process is also shown in Figure 2.

Example 3.

It may be comforting to consider the case where x is distributed normally. That is,

$$p_o = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

then,

$$P_o(w) = e^{-\frac{\sigma^2 w^2}{2}}$$

Therefore,

$$\mathcal{F}\left[P\left\{(x_t - x_o e^{-\theta t}) \mid x_o\right\}\right] = \frac{e^{-\frac{\sigma^2 w^2}{2}}}{e^{\left(-\frac{\sigma^2}{2} \cdot w^2 e^{-2\theta t}\right)}}$$

$$= e^{-\frac{\sigma^2 w^2}{2}(1-e^{-2\theta t})}$$

Taking the inverse of this expression yields

$$P\left\{(x_t - x_o\, e^{-\theta t}) \mid x_o\right\} = \frac{1}{\sigma\sqrt{2\pi(1-e^{-2\theta t})}}\; e^{-\frac{x_t^2}{2\sigma^2(1-e^{-2\theta t})}}$$

$$P\left\{x_t \mid x_o\right\} = \frac{1}{\sigma\sqrt{2\pi(1-e^{-2\theta t})}}\; e^{-\frac{(x_t - x_o e^{-\theta t})^2}{2\sigma^2(1-e^{-2\theta t})}}$$

This result is sketched in Figure 3, and is referred to as the Uhlenbech Process.

Figure 2. The first conditional probability density function $p_1(x)$ for a Markov process with a primary distribution $p_o(x) = \frac{\theta}{2} e^{-\theta|x|}$ and normalized autocorrelation function $\phi(\tau) = e^{-u|\tau|}$. $p_1(x) = e^{-2u\tau}\delta(x-x_o e^{-u\tau}) + (1-e^{-2u\tau})\frac{\theta}{2} e^{-\theta(x-x_o e^{-\tau})}$ gives the distribution of the variable x at $t = \tau$ given that $x = x_o$ at time $t = 0$

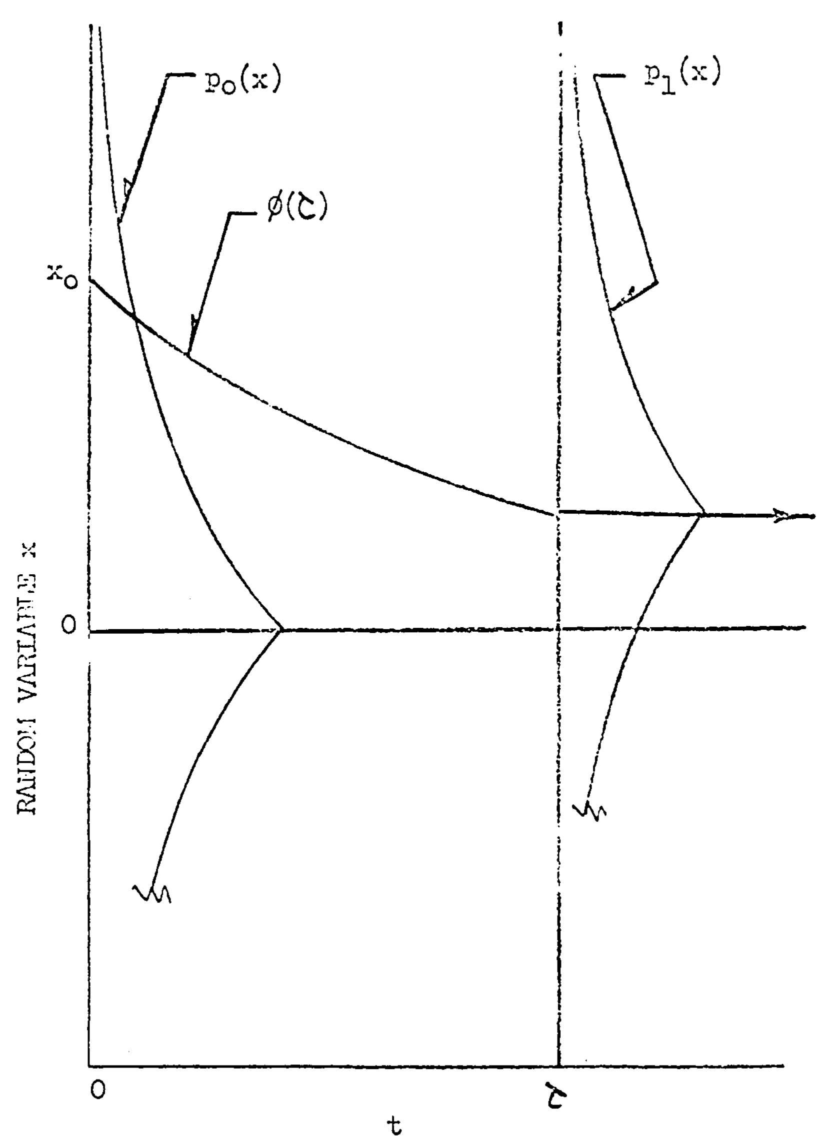
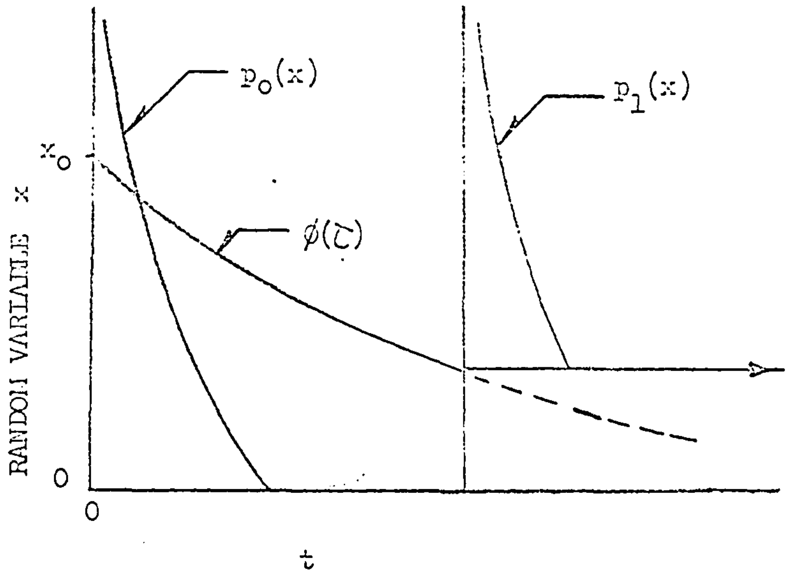
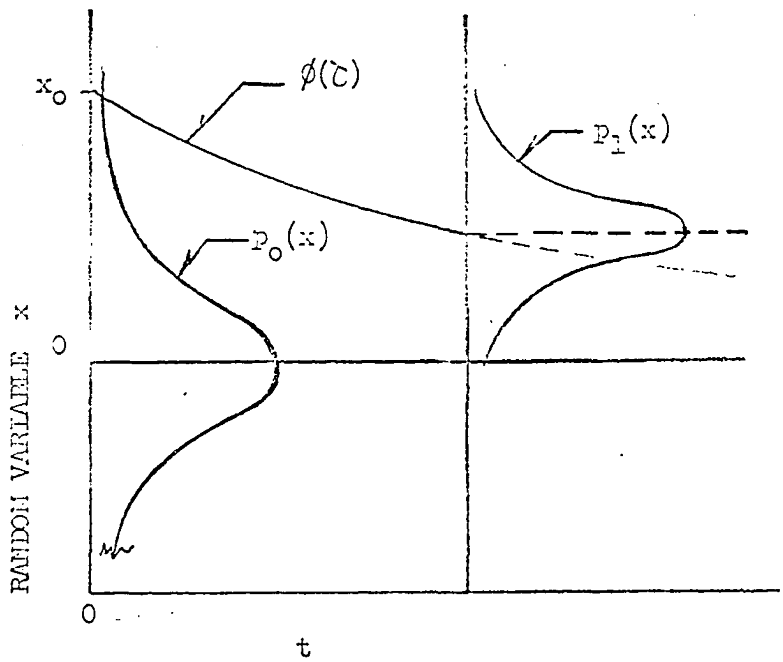

p₀(x)
p₁(x)
ø(τ)
x₀
0
RANDOM VARIABLE x
0
τ
t

Figure 3. Two more examples of $p_1(x)$, the distribution of $t = \lambda$ given $x = x_o$ at $t = C$, and the associated primary distribution $p_o(x)$ for each process



RANDOM VARIABLE x
t
0
0
x₀
p₀(x)
p₁(x)
φ(τ)



RANDOM VARIABLE x
t
0
0
x₀
p₀(x)
p₁(x)
φ(τ)

D. The Axis-Crossing Problem.

Using the conceptual model described above, it is possible to predict various quantities associated with the times at which process value crosses a specified level. The first quantity to be calculated is the probability that the function was above a specified level for a time duration t when the location of the interval is randomly selected. For simplicity, it will be assumed that the specified level is zero; for problems where that is not in fact the case, the probability density function can be redefined by translation so that zero will be the desired level for crossing.

To begin, the probability that the function will be above zero for one pulse is stated as

$$P\left\{x_1 > 0\right\} = \int_0^{\infty} p_o(x_1)\ dx_1$$

where x_1 denotes the "first pulse interval". Now, the probability that the function will be positive over two successive intervals is

$$P\left\{x_1, x_2 > 0\right\} = \int_o \int_o p(x_2, x_1)\ dx_1\ dx_2$$

where $p(x_2, x_1)$ represents the two-dimensional distribution of the values of two successive pulses. This can be restated as

$$P\left\{x_1, x_2 > 0\right\} = \int_o \left[\int_o p_1\left\{x_2 | x_1\right\} p_o(x_1) dx_1 \right] dx_2$$

where $p\left\{x_2 | x_1\right\}$ is the probability density function for the second pulse amplitude given the value of the first pulse amplitude. Therefore

$$P\left\{x_2 \mid x_1 > 0\right\} = \int_0^{\infty} p_1\left\{x_2 \mid x_1\right\} p_o(x_1)\,dx_1$$

Equation 6 shows that for this model, $p\left\{x_2 \mid x_1\right\}$ is a function of $x_2 - \alpha x_1$, where $\alpha = e^{-\theta \Delta t}$. Denote this first conditional probability function as $p_{2|1}(x_2 - \alpha x_1)$. Therefore

$$P\left\{x_2 \mid x_1 > 0\right\} = \int_0^{\infty} p_{2|1}(x_2 - \alpha x_1)\; p_o(x_1)\; dx_1 \tag{11}$$

This can be written in the form of a convolution integral as

$$P\left\{x_2 \mid x_1 > 0\right\} = \int_{-\infty}^{\infty} p_{2|1}(x_2 - \alpha x_1)\; p_o^{+}(x_1)\; dx_1$$

where $p_o^{+}(x_1) = p_o(x_1) \qquad x_1 > 0$

$\qquad\qquad\quad = 0 \qquad\qquad x_1 \leq 0$

Taking the transform of both sides gives

$$P_{2|1>0}(w) = P_o^{+}(\alpha w)\; P_{2|1}(w) \tag{12}$$

Now, $P_{2|1}(w) = P_z(w)$ as defined in Equation 5 since $p_{2|1}(x_2)$ describes the distribution of x_2 translated by a value αx_1. From Equation 5:

$$P_z(w) = \frac{P_o(w)}{P_o(\alpha w)}$$

Then

$$P_{2|1>0}(w) = P_o^{+}(\alpha w)\; \frac{P_o(w)}{P_o(\alpha w)} \tag{13}$$

noting again that $P_o^{+}(w)$ is the Fourier transform of the positive portion of the primary distribution. The inverse of this function, integrated from zero to infinity would yield the probability that the function was positive for two successive pulses.

Continuing the same approach for three successive pulses one obtains:

$$P\left\{x_1, x_2, x_3 > 0\right\} = \int_0^\infty \int_0^\infty \int_0^\infty p(x_1, x_2, x_3)\, dx_1 dx_2 dx_3$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty p_2\left\{x_3 \mid x_2, x_1\right\} \; p_1\left\{x_2 \mid x_1\right\} \; p_o(x_1) dx_1 dx_2 dx_3 \qquad (14)$$

Note that for the processes described by this model

$$p_2\left\{x_3 \mid x_2, x_1\right\} = p_2\left\{x_3 \mid x_2\right\} = p_1(x_3 - \alpha x_2) \qquad (15)$$

Therefore Equation 14 can be written as

$$\int_0^\infty \left\{ \int_0^\infty p_1\left\{x_3 \mid x_2\right\} \left[\int_0^\infty p_1\left\{x_2 \mid x_1\right\} \; p_o(x_1) dx_1 \right] dx_2 \right\} dx_3$$

Then by definition

$$P\left\{x_1, x_2, x_3 > 0\right\} = \int_0^\infty p\left\{x_3 \mid x_2, x_1 > 0\right\} dx_3 = \int_0^\infty p_{3 \mid 2,1 > 0}(x_3)\, dx_3$$

Therefore

$$p_{3 \mid 2,1 > 0}(x_3) = \int_0^\infty p_1(x_3 - \alpha x_2)\; p_{2 \mid 1 > 0}(x_2)\, dx_2$$

Taking the transform of both sides yields;

$$P_{3|2,1>0}(w) = P^{+}_{2|1>0}(\alpha w) \quad P_{2|1}(w)$$

$$= \left[\frac{P_o^{+}(\alpha w) \; P_o(w)}{P_o(\alpha w)} \right]^{+} \frac{P_o(w)}{P_o(\alpha w)} \tag{16}$$

In the same manner, the probability that four successive pulses are positive is

$$P\left\{x_1, x_2, x_3, x_4 > 0\right\} = \int_0^{\infty} P_{x_4 | x_3, x_2, x_1 > 0}(x_4) \; dx_4$$

where

$$P_{4|3,2,1 \; 0}(w) = \left[\left(\frac{P_o^{+}(\alpha w) \; P_o(w)}{P_o(\alpha w)} \right)^{+} \frac{P_o w}{P_o(\alpha w)} \right]^{+} \frac{P_o w}{P_o(\alpha w)} \tag{17}$$

Thus, in principle, the characteristic function for the nth pulse distribution, given that the previous n-1 pulses were positive can be obtained by continuing this procedure. The integration of that function from zero to infinity will yield the probability that the function was positive for n successive pulses.

This process does not produce a convenient closed form solution except for some special cases, one of which will be shown in a following example. It does, however, represent a repetitive convolution process which could be programmed into a high speed machine.

Example 4.

Consider a process with probability density function

$$p_o(x) = \beta/2 \; e^{-\beta |x|}$$

and normalized exponential autocorrelation function:

$$\phi(t) = e^{-\theta |t|}$$

The model described in 3.1 can be used to represent this process to within any arbitrary error.

Then

$$P_{(o)}(w) = \frac{\beta^2}{\beta^2 + w^2}$$

and from Equation 13.

$$P_{2|1>0}(w) = P_o^{+}(\alpha w)\,\frac{P_o(w)}{P_o(\alpha w)}$$

$$= \left(\frac{\beta/2}{\alpha jw + \beta}\right)\left(\frac{\beta^2}{\beta^2 + w^2}\right)\left(\frac{\beta^2 + \alpha^2 w^2}{\beta^2}\right)$$

$$= \beta/2 \;\frac{(\beta - jw\alpha)}{\beta^2 + w^2}$$

$$= \beta/2\,\frac{\left(\frac{1-\alpha}{2}\right)}{(\beta - jw)} + \beta/2\,\frac{\left(\frac{1+\alpha}{2}\right)}{(\beta + jw)} \qquad (18)$$

The probability density function for the second pulse, therefore, given that the first was positive is given by the inverse of this equation:

$$P_{2|1>0}(x_2) = \frac{(1+\alpha)}{4}\; e^{-\beta x_2} \quad x_2 > 0$$

$$= \frac{(1-\alpha)}{4}\; e^{+\beta x_2} \quad x_2 < 0 \qquad (19)$$

Note that the positive portion of this function differs from the positive portion of the primary distribtuion only by a constant. That is

$$p^{+}_{2|1>0}(x) = \frac{(1+\alpha)}{2} \; p_o^{+}(x)$$

Therefore

$$P^{+}_{2|1>0}(w) = \frac{(1+\alpha)}{2} \; P_o^{+}(w)$$

Continuing to include three pulse intervals,

$$\left[P^{+}_{2|1>0}(\alpha w) \; \frac{P_o(w)}{P_o(\alpha w)} \right]^{+} = \frac{(1+\alpha)}{2} \left[P_o^{+}(\alpha w) \; \frac{P_o(w)}{P_o(\alpha w)} \right]^{+}$$

which, by Equation 18 is given by

$$\frac{(1+\alpha)}{2} \left[\frac{(1+\alpha)}{2} \; P_o^{+}(w) \right]$$

Therefore, in general, the Fourier transform for the positive portion of the probability density function for the Nth pulse amplitude given that the previous N-1 pulse were positive is given by

$$P_{N\,|\,(N-1,N-2---)\;\;0}(w) = \left[\frac{1+\alpha}{2} \right]^{N-1} P_o^{+}(w) \qquad (20)$$

Now let $\alpha = e^{-\theta\, t}$, and define an interval $t = N\Delta t$, letting Δt go to zero in the limit as t remains fixed. Then, the probability density function for the process at t is given all values between 0 and t, has a transform:

$$\lim_{\Delta t \to 0} \left[\frac{1 + e^{-\theta \Delta t}}{2} \right]^{t/\Delta t - 1} P_o^+(w)$$

Note as $\Delta t \to 0$, $(t/\Delta t - 1) \to t/\Delta t$. Then set

$$X = \left[\frac{1 + e^{-\theta \Delta t}}{2} \right]^{t/\Delta t}$$

$$\ln X = t \left[\frac{\ln\left(\frac{1+e^{-\theta \Delta t}}{2}\right)}{\Delta t} \right]$$

And

$$\lim_{\Delta t \to 0} \left[\ln X \right] = t \lim_{\Delta t \to 0} \frac{\ln\left(\frac{1+e^{-\theta \Delta t}}{2}\right)}{\Delta t}$$

This is of the form zero over zero, but by L'Hospital's rule,

$$\lim_{\Delta t \to 0} \left[\ln X \right] = t \left[\frac{\frac{-\theta e^{-\theta \Delta t}}{2}}{\left(\frac{1+e^{-\theta \Delta t}}{2}\right)} \right]$$

$$= - \frac{\theta t}{2}$$

Therefore,

$$\lim_{\Delta t \to 0} X = e^{\frac{-\theta t}{2}}$$

From Equation 20

$$P_{N \mid (N-1, N-2---) > 0}(w) = e^{\frac{-\theta t}{2}} P_o^+(w) \qquad (21)$$

The integral from zero to infinity of the inverse of this expression is

$$\int_0^\infty e^{\frac{-\theta t}{2}} \beta/2 \, e^{-\beta x} \, dx$$

$$= \frac{e^{-\frac{\theta t}{2}}}{2} = P\{x > 0 \text{ over interval } t\} \tag{22}$$

Since the process is symmetrical about zero, the probability that the function will be negative over an interval t must be the same as the probability that it will be positive. Therefore;

$$P\{\text{no crossing over interval } t\} = P\{x > 0 \text{ over } t\} + P\{x < 0 \text{ over } t\}$$
$$= e^{-\frac{\theta t}{2}} \tag{23}$$

It follows that the probability of at least one crossing in an interval t is given by

$$1 - e^{-\frac{\theta t}{2}} \tag{24}$$

Continuing, the probability of <u>exactly</u> <u>one</u> crossing in an interval t will be calculated. To do this, consider the interval t consists of n pulse intervals. Further, define the following events:

Event 1 = The function is positive for the first j pulses of the interval t. $(j < n)$

Event 2 = The $(j+1)^{st}$ pulse is positive.

Event 3 = The $(j+1)^{st}$ pulse is negative.

Event 4 = The $(j+2)$ through nth terms are negative.

Then $P\{3,1\}$ is the probability that the function is positive for j intervals and then crosses the axis between intervals j and j + 1. From Equation 23,

$$P\{1,2\} = e^{-\frac{\theta}{2}(j+1)\Delta t}$$

$$P\{1\} = e^{-\frac{\theta}{2}(j\Delta t)}$$

Then since

$$P\{1,2\} = P\{2|1\}\, P\{1\}$$

$$P\{2|1\} = \frac{P\{1,2\}}{P\{1\}} = e^{-\frac{\theta}{2}\Delta t}$$

Further,

$$P\{2|1\} + P\{3|1\} = 1$$

Therefore

$$P\{3|1\} = 1 - P\{2|1\}$$

$$= 1 - e^{-\frac{\theta}{2}\Delta t}$$

But

$$P\{3,1\} = P\{3|1\}\ P\{1\}$$

$$= \frac{e^{-\frac{\theta}{2}(j\Delta t)}}{2}\left[1-e^{-\frac{\theta}{2}\Delta t}\right]$$

Now, from Equation 18 it is known that the form of the negative portion of the probability density function for the magnitude of a pulse, given the previous pulse was positive, is of the same exponential order and form as the primary distribution. Therefore, the probability that the function will be negative from the $(j+1)^{st}$ pulse through the nth pulse given that it was positive for the first j intervals can be calculated in the same manner. That is;

$$P\{1,3,4\} = P\{4|1,3\} \cdot P\{1,3\}$$

$$= \left[e^{-\frac{\theta}{2}(n-j-1)\Delta t}\right]\left[\frac{e^{-\frac{\theta}{2}(j\Delta t)}}{2}\left(1-e^{-\frac{\theta}{2}\Delta t}\right)\right]$$

$$= \frac{e^{-\frac{\theta}{2}(n-1)}}{2}\left(1-e^{-\frac{\theta}{2}\Delta t}\right)$$

This is the probability, that in an interval t, the function will be positive for exactly j intervals, cross the axis between the jth and $(j+1)^{st}$ interval and be negative for the remainder of the duration t. Note that it is independent of j, which indicates that the one crossing in interval t may occur with equal probability between any of the n intervals. There are n-1 such locations, so that the probability of there being exactly one crossing from positive to negative in an interval t is

$$(n-1)\,\frac{e^{-\frac{\theta}{2}(n-1)}}{2}\,(1-e^{-\frac{\theta}{2}\Delta t})$$

From symmetry the probability of there being exactly one crossing would be twice this value.

Since there are n intervals of Δt in t

$$n = \frac{t}{\Delta t}$$

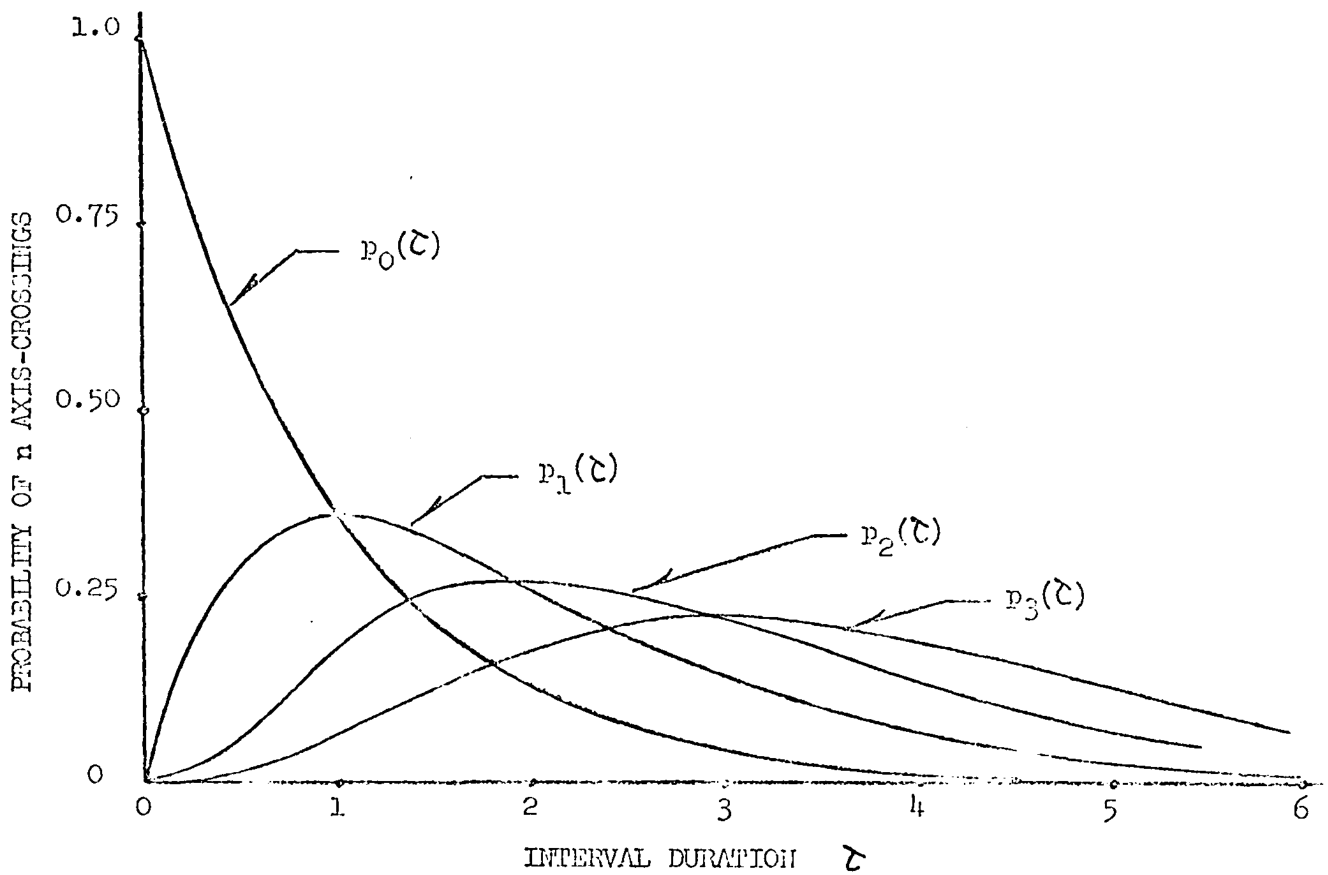
and the probability of there being exactly one crossing in an interval t, as $\Delta t \to 0$ is

$$\lim_{\Delta t \to 0} \left\{ \left(\frac{t}{\Delta t} - 1\right) \left[\frac{e^{-\frac{\theta}{2}\left(\frac{t}{\Delta t} - 1\right)}}{2} \right] (1-e^{-\frac{\theta}{2}\Delta t}) \right\}$$

$$= \left(\frac{\theta t}{2}\right) e^{-\frac{\theta t}{2}} \quad \text{(See Figure 4)} \qquad (25)$$

Similarly, the probability of exactly two crossings can be calculated, but it is simple to note that Equations 23 and 25 are the same form as the probability for zero or one event in a Poisson process. For such a process;

Figure 4. Shows $p_n(\lambda)$, the probability that exactly n crossings occurring in a randomly chosen interval of duration λ, for the process described in Example 4


PROBABILITY OF n AXIS-CROSSINGS
1.0
0.75
0.50
0.25
0
$p_0(\tau)$
$p_1(\tau)$
$p_2(\tau)$
$p_3(\tau)$
0
1
2
3
4
5
6
INTERVAL DURATION τ

$$P\left\{n \text{ crossings}\right\} = p_n(t) = \left(\frac{\theta t}{2}\right)^n \frac{1}{n!} e^{-\frac{\theta t}{2}} \qquad (26)$$

Here then, is a continuous Markov process whose zero crossing durations form a Poisson process.

The "apparent period" or average axis crossing duration can be determined by first calculating the average value of t_n, the duration in which n crossings will occur. Then since there are n+1 durations for n crossings in a given interval, the average time between crossings is given by (Avg. t_n)/(n+1).

$$\text{Avt } t_n = \int_o^\infty t \frac{\left(\frac{\theta t}{2}\right)^n}{n} e^{-\frac{\theta t}{2}}$$

$$= \frac{\left(\frac{\theta}{2}\right)^n}{n!} \int_o^\infty (t)^{n+1} e^{-\frac{\theta t}{2}} dt$$

$$= (n+1) \left(\frac{\theta}{2}\right) \qquad (27)$$

Therefore, the average duration between crossings is $\left(\frac{\theta}{2}\right)$.

E. <u>The Filter Problem</u>.

The problem is as follows; if a non-gaussian random process is the input to a linear filter, what is the probability density function of the output? In this section this problem is treated only with respect to processes representable by the first-order model described in section II-A. In a later section, the same problem is posed for a higher order model.

Assume a process x(t) is the input to a filter with an impulse response of h(t) and a resulting output y(t).

x(t) is assumed to be made up of a series of rectangular pulses; then y(t) can be written as the convolution of the input x(t) with the impulse response h(t):

$$y(t) = \int_{o}^{t} h(u)\ x(t-u)du \tag{28}$$

Since x(t) is made up of a series of rectangular pulses of duration t, this can be written as;

$$y_n = x_n \int_{o}^{\Delta t} h(t)dt + x_{n-1} \int_{\Delta t}^{2\Delta t} h(t)dt \text{ -----} \tag{29}$$

where x_i = magnitude of ith pulse. Note that $y(t) = y_n$ implies y(t) is being evaluated at the end of the nth input pulse. For convenience define:

$$K_1 = \int_{o}^{t} h(t)\ dt,$$

$$K_2 = \int_{\Delta t}^{2\Delta t} h(t)\ dt$$

$$K_n = \int_{\Delta(n-1)\Delta t}^{n\Delta t} h(t)\ dt \tag{30}$$

Then

$$y_n = K_1 x_n + K_2 x_{n-1} + K_3 x_{n-2} + \text{ -----} \tag{31}$$

In an ensemble sense, x_n, x_{n-1}, x_{n-2} ----- are random variables from the same sample space related by Equation 1, and K_1, K_2, K_3 ---- are constants.

From Equation 2;

$$x_n = z_n + \alpha z_{n-1} + \alpha^2 z_{n-2} + ---$$

$$x_{n-1} = z_{n-1} + \alpha z_{n-2} + \alpha^2 z_{n-3} ---$$

Therefore

$$y_n = K_1(z_n + \alpha z_{n-1} + \alpha^2 x_{n-2} ---)$$
$$+ K_2(z_{n-1} + \alpha z_{n-2} + \alpha^2 z_{n-3} ---)$$
$$+ K_3(z_{n-2} + \alpha z_{n-3} ---)$$

which can be rearranged as

$$y_n = z_n(K_1)$$
$$+ z_{n-1}(K_2 + \alpha K_1)$$
$$+ z_{n-2}(K_3 + \alpha K_2 + \alpha^2 K_1) \quad (32)$$

Letting $K_1 = a$, $K_2 + \alpha K_1 = b$, $K_3 + \alpha K_2 + \alpha^2 K = c$ etc.,

$$y_n = az_n + bz_{n-1} + cz_{n-2} ---. \quad (33)$$

Then the characteristic functions are related by

$$Y(w) = Z(\alpha w)Z(bw)Z(cw) --- \quad (34)$$

Thus, in principle, the characteristic function of the probability density function for the output of <u>any</u> linear filter can be determined given an input which can be approximated by this conceptual model. Mechanically one needs only to determine the constants a, b, c --- in Equation 33, carry out the multiplication, and take the inverse transform. The number of constants required will, of course, depend upon the specific problem, but in general, choosing a relatively small Δt will decrease the error of the representation both at the input and output of the filter, but will greatly increase the computation required to obtain an answer. This technique is illustrated by calculating the output distribution of three different filters with the same non-Gaussian process as inputs.

Example 5.

Consider the continuous process with a probability density function

$$p(x) = \beta e^{-\beta x} \qquad x > 0$$
$$= 0 \qquad x < 0$$

an autocorrelation function

$$\phi(\tau) = e^{-|\tau|}$$

as the input to a simple R-C low-pass filter with an impulse response

$$h(t) = 1/RC\ e^{-t/RC}$$

With the assistance of a digital computer, the output distributions were calculated for values of RC = 1, RC = 3, and RC = 6. The essentials of the case RC = 3 is outlined below, and the plotted results for all three cases are shown in Figure 4. For RC = 3 a value of $\Delta t = 1$ was chosen, thereby implying

$$\alpha = e^{-1} = 0.368$$

Somewhat arbitrarily, twelve values of k were calculated. The relative insignificance of several of the resulting terms confirmed that this was more than adequate.

They are;

$$K_1 = \int_o^1 1/3\ e^{-t/3} dt = 0.284$$

$$K_2 = \int_o^2 1/3\ e^{-t/3} dt = 0.203$$

$K_3 = 0.145$

$K_4 = 0.104$

$K_5 = 0.074$

$K_6 = 0.053$

$K_7 = 0.038$

$K_8 = 0.027$

$K_9 = 0.019$

$K_{10} = 0.014$

$K_{11} = 0.010$

$K_{12} = 0.007$

Following from Equation 33,

$a = K_1 = 0.284 = 1/3.65$

$b = \alpha K_1 + K_2 = 0.304 = 1/3.26$

$c = 0.258 = 1/3.88$

$d = 1/5.02$

$e = 1/6.8$

$f = 1/9.3$

h = 1/17.9

i = 1/25.0

j = 1/33.4

k = 1/47.5

l = 1/66.5

Therefore, from Equation 5,

$$z(s) = \frac{\alpha s + \beta}{s + \beta}$$

and from Equation 34,

$$Y(s) = \left(\frac{\alpha as + \beta}{\alpha s + \beta}\right)\left(\frac{\alpha bs + \beta}{bs + \beta}\right) \text{-----------} \left(\frac{\alpha ls + \beta}{ls + \beta}\right)$$

$$= \left(\frac{\alpha s + \beta/a}{s + \beta/a}\right)\left(\frac{\alpha s + \beta/b}{s + \beta/b}\right) \text{-----------} \left(\frac{\alpha s + \beta/l}{s + \beta/l}\right)$$

$$= \alpha^{12} + \frac{A}{s + \beta/a} + \frac{B}{s + \beta/b} \text{------------} + \frac{L}{s + \beta/l}$$

Calculation of the constants A,B,C, -----L provided

$A = -1.545 \times 10^3$

$B = 4.502 \times 10^2$

$C = 1.166 \times 10^3$

$D = -73.41$

$E = 2.43$

$F = 2.569 \times 10^{-3}$

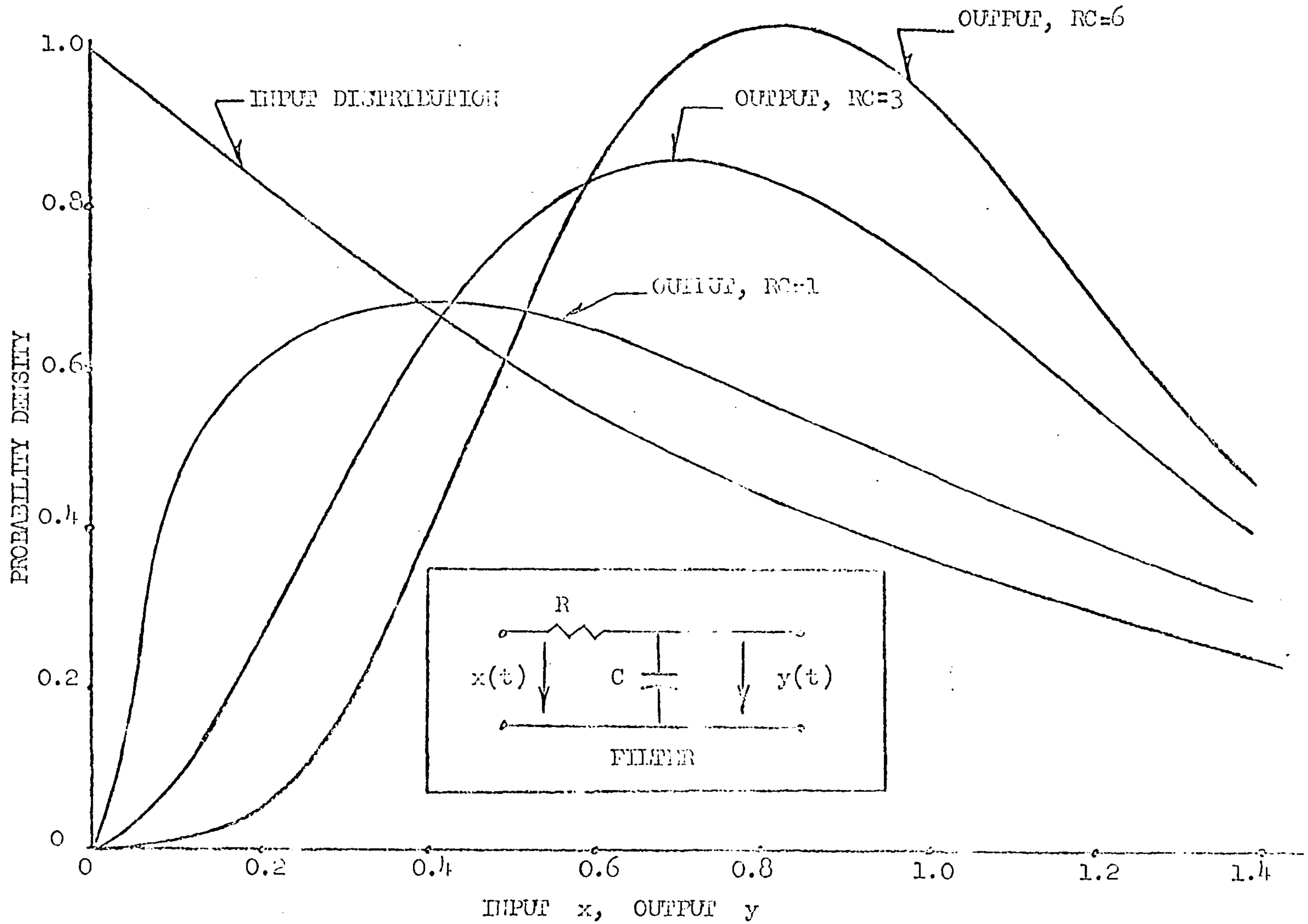
$L = 4.601 \times 10^{-6}$

The term α^{12} must be ignored since these equations are really truncated approximations of an infinite product. Therefore, the output can be written;

$p(y) = -1.545 \times 10^{3} \ e^{-3.65} + 4.502 \times 10^{2} \ e^{-3.26} \ ------ + 4.601 \times 10^{-6} \ e^{-66.5}$

This function, the results for the other two cases, and the common input distribution are shown in Figure 4.

Figure 5. Shows the output distributions for three different filters with the same input, also shown. The three filters are simple low-pass filters with impulse responses $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}}$ for values of RC equal to 1, 3, and 6.


OUTPUT, RC=6
INPUT DISTRIBUTION
OUTPUT, RC=3
OUTPUT, RC=1
PROBABILITY DENSITY
1.0
0.8
0.6
0.4
0.2
0
R
x(t)
C
y(t)
FILTER
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
INPUT x, OUTPUT y

III. MODEL FOR A SECOND-ORDER PROCESS

A. The Difference Equation.

As in Section II, assume a random process generated by a series of adjacent rectangular pulses of duration T. In this case, however, the magnitudes of the pulses are related by;

$$x_n = z_n + \alpha_1 x_{n-1} + \alpha_2 x_{n-2} \tag{35}$$

where again,

x_n = magnitude of the nth pulse

x_{n-1} = magnitude of the first pulse preceding the nth pulse

z_n = an independent random variable

α_1, α_2 = constants.

Equation 35 can also be written as

$$x_n = a_o z_n + a_1 z_{n-1} + a_2 z_{n-2} + a_3 z_{n-3} \cdots \tag{36}$$

where

$a_o = 1$

$a_1 = \alpha_1$

$a_2 = \alpha_1^2 + \alpha_2$

$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2}$.

This process will be shown to represent an approximation of a fairly wide range of second order processes for various choices of α_1 and α_2.

B. The Autocorrelation Function.

Since the simulation of a process with a known autocorrelation function requires the proper choice of constants α_1 and α_2, it is desirable to have a systematic method to make this choice. Because the process defined is essentially a sampled-data sequence, many of the techniques utilized in sampled data systems are directly applicable, particularly in regard to the autocorrelation function.

Starting with the difference equation 35,

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + z_n$$

and taking its Z-transform, one obtains;

$$X(z) = Z(z) \frac{z^2}{z^2 - \theta_1 z - \alpha_2}$$

One can consider the process X(z) as the output of a filter with a z-transform transfer function

$$G(z) = \frac{z^2}{z^2 - \alpha_1 z - \alpha_2} \tag{38}$$

and an input process z(z). The relation between the autocorrelation function of the input process z and the output process X is given by:

$$\Phi_{xx}(z) = G(z)G(z^{-1}) \Phi_{zz}(z)$$

In the model presented, the magnitudes of z's are independent of each other, so that on a normalized basis,

$$\Phi_{zz}(z) = 1$$

and therefore

$$\Phi_{xx}(z) = G(z)G(z^{-1}) \tag{39}$$

Thus, given the difference equation 35, one can determine the value of the autocorrelation function at integral values of T. Since the model process is a sequence of rectangular pulses, a straight line connection of these points will represent the autocorrelation function for the model.

Similarly, if one knows the autocorrelation function of a process to be approximated, the constants of the difference equation can be determined by factoring $\Phi_{xx}(z)$ in to $G(z)G(z^{-1})$.

Example 6.

Assume the continuous process to be approximated can be considered as the equivalent of the output of a filter $G(s) = \frac{w_o}{(s+a)^2+w_o^2}$ with a white, but not necessarily gaussian, random process as an input. If $w_o = 1$, $a = 0.22$;

$$G(s) = \frac{1}{(s+.22)^2+1}$$

and

$$g(t) = e^{-0.22t} \sin t$$

The z-transformation of g(t) is written;

$$G(z) = e^{-0.22T} \sin T \quad \frac{z}{z^2-(2e^{-0.22T}\cos T)\ z+e^{-0.44T}}$$

Comparison with Equation 38 shows that

$$\alpha_1 = 2e^{-0.22T} \cos T$$

$$\alpha_2 = -e^{-0.44T}$$

If T is chosen to be $\frac{\pi}{4}$:

$\alpha_1 = 1.18$

$\alpha_2 = -0.707$

Having determined these constants, $\phi_{xx}(z)$ can be plotted by using Equation 39, or as a mechanically easier alternative, it can be shown that $\phi_{xx}(nT)$ obey the following difference equation and initial conditions.

$$(1) \quad \phi_{xx}(T) = \frac{\alpha_1}{1 - \alpha_2} \phi_{xx}(0)$$

$$(2) \quad \phi_{xx}(nT) = \alpha_1 \phi_{xx}(n-1)T + \alpha_2 \phi_{xx}(n-2)T \qquad \text{for } n > 1 \quad (40)$$

Figure 6 shows the autocorrelation function for the model process as compared to the autocorrelation function for the continuous process. Figure 7 shows some other possible plots of $\phi(nT)$ for various combinations of α_1 and α_2.

C. Conditional Probability Density Function.

In order to eventually solve the filter problem for the second order process it is necessary to develop an expression which will provide the distribution of the "generating" random variable z given the distribution of the process to be approximated. To do this, manipulation of some conditional probability density functions is required and the terms defined as follows will be employed.

(1) $P\{X_n\} = p_x(x_n)$ = primary distribution of the random process x.

(2) $P\{Z_n\} = p_z(z_n)$ = distribution of the generating variable z.

Figure 6. Compares the autocorrelation function of the output of a filter having a transfer function $G(s) = \frac{1}{(s + 0.22)^2 + 1}$ and a white noise input, with the autocorrelation function for a rectangular-pulse model process

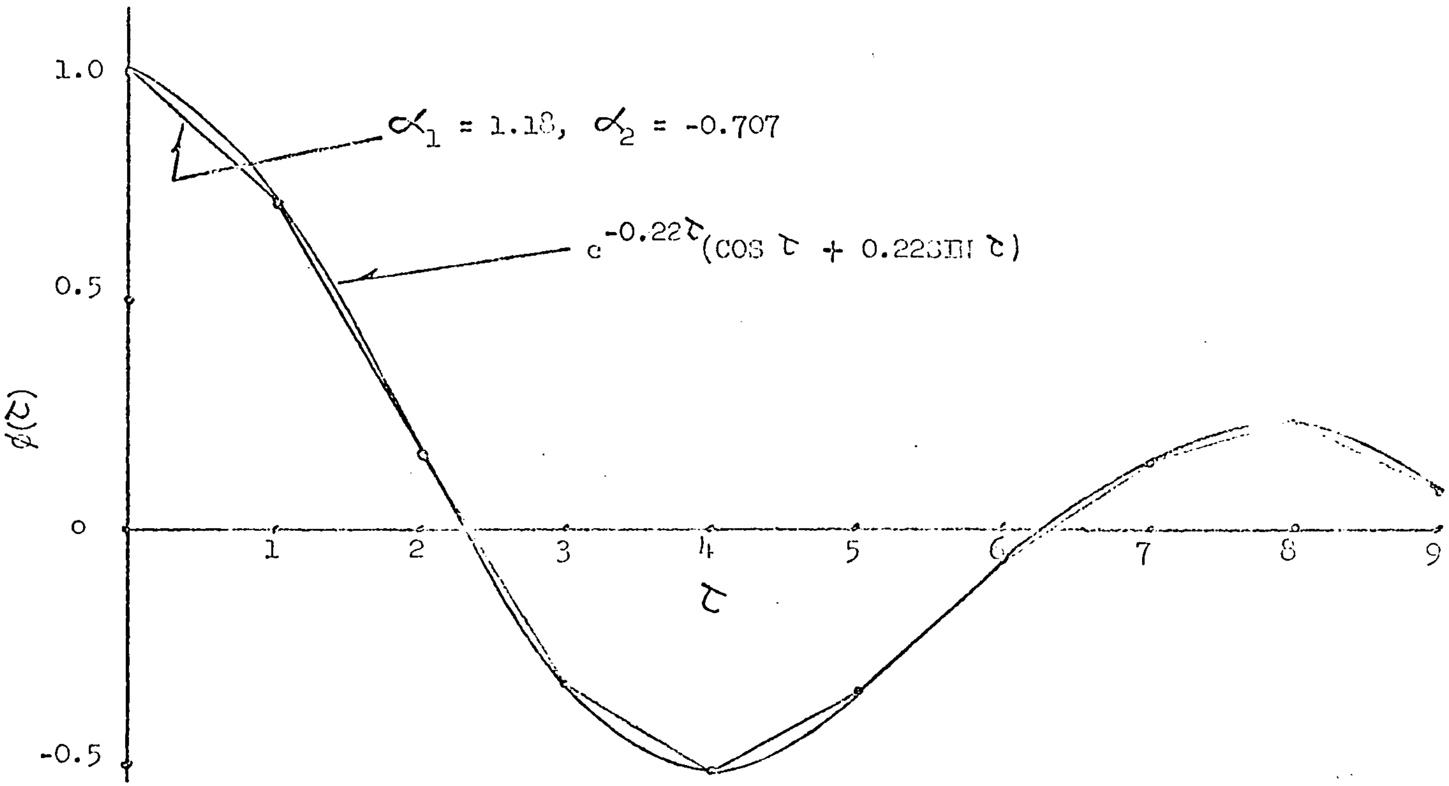
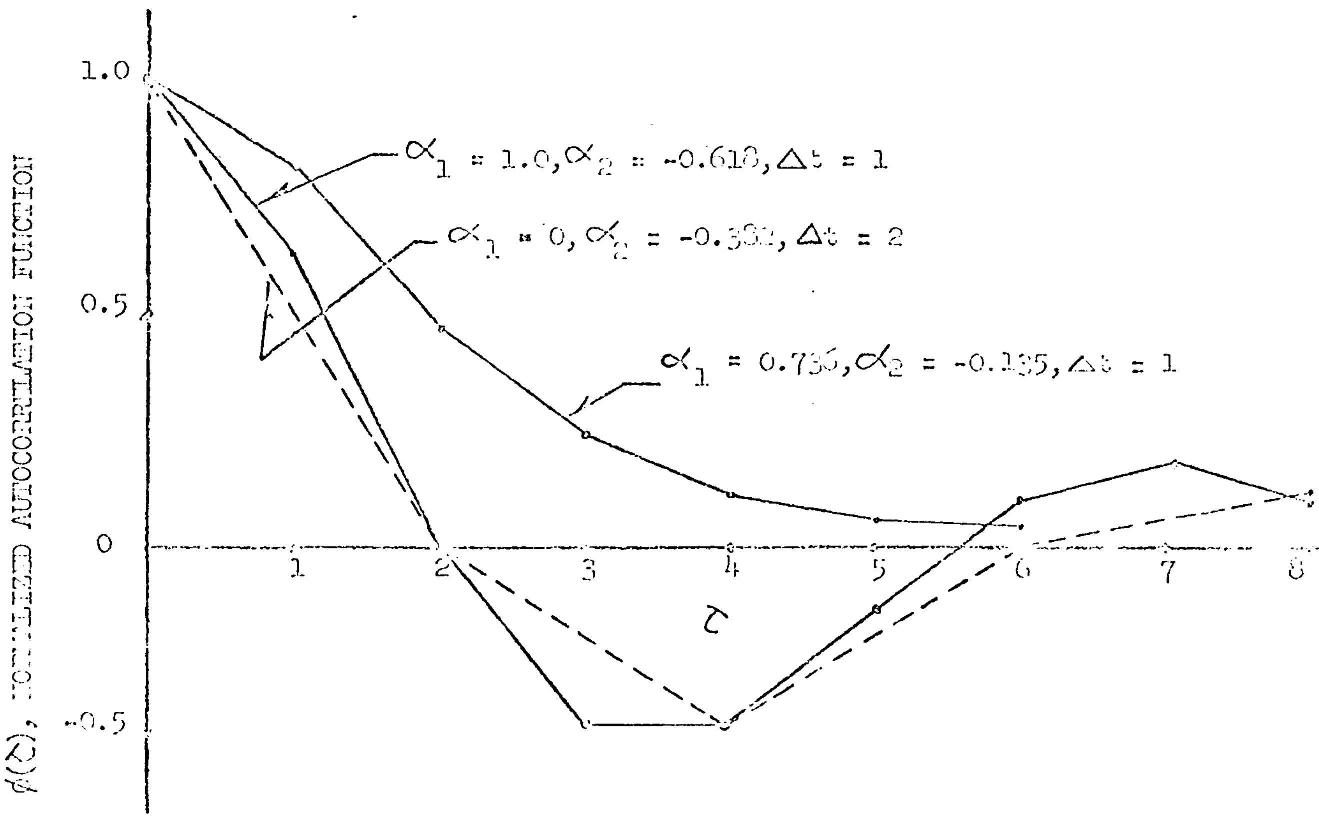

α1 = 1.18, α2 = -0.707
e-0.22τ(COS τ + 0.22SIN τ)
1.0
0.5
0
-0.5
φ(τ)
1
2
3
4
5
6
7
8
9
τ

Figure 7. Shows three different examples of autocorrelation functions for the second-order process. The values of α_1 and α_2 for each example are labeled in the figure



φ(τ), NORMALIZED AUTOCORRELATION FUNCTION
1.0
0.5
0
-0.5
1
2
3
4
5
6
7
8
τ
α1 = 1.0, α2 = -0.618, Δt = 1
α1 = 0, α2 = -0.382, Δt = 2
α1 = 0.736, α2 = -0.135, Δt = 1

(3) $P\left\{X_n | X_{n-j}\right\} = f_j(X_n - k_j X_{n-j})$ gives the conditional distribution of X_n if X_{n-1} is defined.

The last equation, stating that the probability density function for the magnitude of the nth pulse, given the value of the $(n-j)^{th}$ pulse, can be written in the form $f_1(x_n - k_j X_{n-j})$ may be an arbitrary assumption in that its uniqueness is not proven here. It is, however, consistent within itself. That is, all conditional probability statements are satisfied and the autocorrelation function is correctly predicted by this assumption.

On these premises then, the statement

$$\int P\left\{x_n | x_{n-1}\right\} \; P\left\{x_{n-1}\right\} \; dx_{n-1} = P\left\{x_n\right\}$$

can be written

$$\int f_1(x_n - k_1 x_{n-1}) \; p_x(x_{n-1}) dx_{n-1} = p_x(x_n)$$

The Fourier transform of this equation yields

$$P_x(k_1 w) \; F_1(w) = P_x(w)$$

or

$$F_1(w) = \frac{P_x(w)}{P_x(k_1 w)} \tag{41}$$

Similarly,

$$P\left\{x_n | x_{n-2}\right\} = \int P\left\{x_n | x_{n-1}, x_{n-2}\right\} \; P\left\{x_{n-1} | x_{n-2}\right\} dx_{n-2}$$

From the process definition, it can be seen that

$$P\left\{x_n | x_{n-1}, x_{n-2}\right\} = p_z(x_n - \alpha_1 x_{n-1} - \alpha_2 x_{n-2})$$

Therefore;

$$f_2(x_n - k_2 x_{n-2}) = \int p_z(x_n - \alpha_1 x_{n-1} - \alpha_2 x_{n-2}) f_1(x_{n-1} - k_1 x_{n-2}) dx_{n-1} \tag{42}$$

$$e^{jw(k_2 X_{n-2})}F_2(w) = e^{jw(\alpha_2 x_{n-2})}P_z(w) \quad e^{jw\alpha_1(k_1 x_{n-2})}F_1(\alpha_1 w)$$

$$= e^{jwx_{n-2}(\alpha_2+\alpha_1 k_1)} P_z(w)F_1(\alpha_1 w) \tag{43}$$

This equation provides two results

(1) $k_2 = \alpha_1 k_1 + \alpha_2$

This is obviously true if $p_x(x)$ is symmetrical about $x = 0$ since then $f_1(x)$, and therefore, $f_2(x)$ must also be symmetrical about zero from Equation 41 and $p_z(z)$ certainly must be symmetrical about zero, if a sum of independent z's yield a symmetrical distribution. It can also be verified for this and other cases on the basis that it is consistent with the autocorrelation function statement. That is:

$$\phi(jT) = \iint x_n x_{n-j} \ P\left\{x_n, x_{n-j}\right\} dx_n dx_{n-j}$$

$$= \iint x_n x_{n-j} \ P\left\{x_n \middle| x_{n-j}\right\} P\left\{x_{n-j}\right\} dx_n dx_{n-j}$$

$$= \iint x_n x_{n-j} f_j(x_n - k_j x_{n-j}) p_x(x_{n-j}) dx_n dx_{n-j}$$

$$= \int x_{n-j} p_x(x_{n-j}) \left[\int x_n f_j(x_n - k_j x_{n-j}) dx_n \right] dx_{n-j}$$

The quantity within the brackets is just the average of the function $f_j(x_n - k_j x_{n-j})$, denoted by $\langle f_j(x_n - k_j x_{n-j}) \rangle$. But,

$$\langle f_j(x_n - k_j x_{n-j}) \rangle = k_j x_{n-j} + \langle f_j \rangle$$

Therefore

$$\phi(jT) = k_j \int x^2_{n-j} p_x(x_{n-j}) dx_{n-j} + \langle f_j \rangle \int x_{n-j} P_x(x_{n-j} dx_{n-j}$$

$$= k_j \langle x^2_{n-j} \rangle + \langle f_j \rangle^2$$

By subtracting out the average value $\langle f_j \rangle^2$ and noting that $\phi(0) = \langle x_n^2 \rangle$, it can be seen that the normalized autocorrelation function must be

$$\phi(jT) = k_j \tag{44}$$

Therefore, from Equation 40

$$k_2 = \alpha_1 k_1 + \alpha_2 k_{(0)}$$
$$= \alpha_1 k_1 + \alpha_2$$
$$= \alpha_1 \phi_{(T)} + \alpha_2$$

(2) Equation 43 also shows that

$$F_2(w) = F_1(\alpha_1 w)\ P_z(w). \tag{45}$$

Continuing the same process to determine the transformation relationship for $P\left\{x_n \middle| x_{n-3}\right\}$;

$$P\left\{x_n \middle| x_{n-3}\right\} = \iint P\left\{x_n \middle| x_{n-1}, x_{n-2}, x_{n-3}\right\}\ P\left\{x_{n-1} x_{n-2} x_{n-3}\right\} P\left\{x_{n-2}\ x_{n-3}\right\} dx_{n-1}, dx_{n-2}$$

noting that

$$P\left\{x_n \middle| x_{n-1}, x_{n-2}, x_{n-3}\right\} = P\left\{x_n \middle| x_{n-1}, x_{n-2}\right\}$$

we have

$$f_3(x_n - k_3 x_{n-3}) = \iint p_z(x_n - \alpha_1 x_{n-1} - \alpha_2 x_{n-2})\ p_2(x_{n-1} - \alpha_1 x_{n-2} - \alpha_2 x_{n-3})$$
$$\times\ f_1(x_{n-2} - k_1 x_{n-3})\, dx_{n-1}\, dx_{n-2}$$
$$= \int f_1(x_{n-2} - k_1 x_{n-3})\, dx_{n-2} \int p_z(x_n - \alpha_1 x_{n-1} - \alpha_2 x_{n-2})$$
$$p_z(x_{n-1} - \alpha_1 x_{n-2} - \alpha_2 x_{n-3})\, dx_{n-1}$$

The Fourier transform of this equation with respect to x_n is

$$e^{jw(k_3 x_{n-3})} F_3(w) = \left[e^{jw\, k_1(\alpha_1^{\,2}+\alpha) x_{n-3}} \; F_1 \quad (\alpha^2{}_1 + \alpha_2) w \right] \cdot \tag{46}$$

$$\left[e^{jw(\alpha_1 \alpha_2 x_{n-3})} \; P_z(\alpha_1 w) \; P_z(w) \right]$$

Therefore

$$k_3 = k_1(\alpha_1^{\,2} + \alpha_2) + \alpha_1 \alpha_2$$

$$= \alpha_1(\alpha_1 k_1 + \alpha_2) + \alpha_2 k_1$$

$$= \alpha_1 k_2 + \alpha_2 k_1 \tag{47}$$

Also,

$$F_3(w) = F_1 \left[(\alpha_1^{\,2} + \alpha_2) w \right] P_z(\alpha_1 w) \; P_z(w) \tag{48}$$

Using Equation 40, this can be written

$$F_3(w) = F_1(a_2 w) \; P_z(a_1 w) \; P_z(w) \tag{49}$$

And in general,

$$F_n(w) = F_1(a_n w) \; P_z(a_{n-1} w) \; --- \; P_z(a_1 w) \; P_z(w). \tag{50}$$

Thus, Equations 44 and 50 define all first conditional probability density functions.

In order to define $P_z(w)$ in terms of $P_x(w)$, one more relationship is necessary;

$$P\{x_n\} = \int P\{x_n | x_{n-2}\} \; P\{x_{n-2}\} dx_{n-2}$$

The transform of this gives,

$$P_x(w) = F_2(w) \; P_x(k_2 w) \tag{51}$$

or

$$F_2(w) = \frac{P_x(w)}{P_x(k_2 w)}.$$

Therefore, using Equations 41, 44, and 45,

$$P_z(w) = \frac{F_2(w)}{F_1(\alpha_1 w)}$$

$$= \frac{P_x(w)}{P_x(k_2 w)} \quad \frac{P_x(\alpha_1 K_1 w)}{P_x(\alpha_1 w)}$$

$$= \frac{P_x(w)}{P_x(\alpha_1 w)} \quad \frac{P_x(\alpha_1 \phi_1 w)}{P_x(\phi_2 w)} \tag{52}$$

This establishes the distribution required for the random variable z in order to produce the prescribed distribution $p_x(x)$ for the process x(z). Note that if $\alpha_1 \phi_1 = \phi_2$, as for a Markov process, Equation 52 agrees with Equation 5,

D. The Filter Problem.

If the second order process model is now introduced as the input of an arbitrary filter H(s), the determination of the distribution of the filter output y(t) is straight forward and follows directly the development of section II-E. From Equation 36,

$$x_n = z_n + a_1 z_{n-1} + a_2 z_{n-2} + \text{---}$$

and from Equation 31,

$$y_n = k_1 x_n + k_2 x_{n-1} + k_3 x_{n-2} \text{ ---}$$

where again

$$k_n = \int_{(n-1)\Delta T}^{n \Delta T} h(t)\, dt$$

Combining these equations

$$y_n = z_n(K_1) + z_{n-1}(K_2 + a_1 K_1) + z_{n-2}(k_3 + a_1 k_2 + a_2 k_3) \text{ ---}$$

$$= C_o z_n + C_1 z_{n-1} + C_2 z_{n-2} \text{ ---}$$

Therefore, the characteristic function for the distribution of the filter output y(t) is given by,

$$P_y(w) = P_z(C_o w)\, P_z(C_1 w)\, P_z(C_2 w) \cdots \qquad (53)$$

Inversion of this product is accomplished as illustrated in Example 5.

IV. ALLOWED DISTRIBUTIONS

Normally, the distribution of a random process would be defined by the underlying cause of the process. For instance, in the case of thermal noise, a diffusion type of differential equation dictated by the physics of the problem requires that the process be normally distributed. For problems where the cause may not be known, or fully understood, it would be desirable to determine what the restrictions are on the form of the probability density functions which can be approximated by these techniques.

Equations 5 and 50 show that the conditional probability density functions are determined from the primary distribution. These equations relating the characteristic functions for the conditional and primary distributions clearly to impose some restrictions. Relevant to this problem, Gnedenko and Kolomogorov (5) examined the properties of "distribution functions of class L". A necessary and sufficient condition that a function p(x) belongs to the class L is that for every $0 < \alpha < 1$,

$$P(w) = P(\alpha w)\ G(w)$$

where P(w) is the characteristic function for p(x) and G(w) is some other characteristic function. Clearly the primary distributions for the models outlined here must belong to the class L. Although a precise distinction between the distributions which can be described by these models and those which cannot has not been determined, the following requirements on the primary distribution are noted here:

a). The characteristic function for the primary distribution must have no zero's along the w axis. This can be intuitively seen from

Equation 5 where $P_z(w)$ is defined at $P_x(w)/P_x(\alpha w)$. If $P_x(w)$ is zero for some w, then $P_x(\alpha w)$ will be zero for some other value of w. If the freedom to choose any pulse duration, i.e. any value of α, is retained, then, in general, the zeros of $P_x(\alpha w)$ will be poles for the expression $P_z(w)$ and the inverse of such a transform could not represent a probability density function.

This restriction eliminates such distributions as;

1). Uniform distribution over a finite range

2). triangular distributions

3). truncated sinusoid,

and many others.

b). The range of the variable Z can not be greater than the range of the process x(t). This is seen from

$$x_n = z_n + a_1 z_{n-1} + a_2 z_{n-2} --- .$$

Clearly x_n can assume at least as wide a range of values as Z_n. This restriction precludes for example

$$p_x(x) = e^{-x} \qquad 0 < x < a$$

$$= 0 \qquad x < 0,\ x > a$$

and many other truncated continuous functions.

c). Of course, both $p_z(z)$ and $p_x(x)$ must be always positive and their integrals over the total range of definition for the problem must equal unity.

V. INPUT DISTRIBUTIONS FROM OUTPUT DISTRIBUTIONS

Equations 5 and 52 give the characteristic equation for the random variable z in terms of the characteristic function for the random process x, for 2 separate difference equations. Thus, in each case, the distribution of z can be determined from a specified distribution of X. It was pointed out that a specific difference equation leads to a specific z-transform filter transfer function G(z), relating the random variable $x_{(z)}$ as the sampled data output of the filter and $Z_{(z)}$ as the sampled data input to the filter. The assumption for this model that successive values of z are assumed to be independent is equivalent to assuming that the filter has a "white noise" input.

Either equations 5 or 52, therefore, will provide a solution to the following problem:

Assume;

1. That the spectrum of the input to a filter is known to be wide with respect to that filter and is essentially "white".
2. That the filter has a z-transform transfer function G(z) which can be derived from the difference equation of one of the three models presented. In fact, the second order difference equation includes the first order case by choosing α_2 equal to zero. Further, note that, the relation between the input and output distributions is unchanged by whether the subscript on z is n, n-1, or n-j. This means that the numerator to G(z) can be z raised to any power. Therefore, any filter with transfer function

$$G(z) = \frac{z^n}{z^2 + \alpha_1 z + \alpha_2}$$

can be considered.

3. That the output distribution is known.

Under these assumptions, either Equation 5 or 52 will give the characteristic function for the input distribution.

Example 7.

Suppose the output of a filter with a transfer function,

$$G(s) = \frac{k}{(s+1)^2}$$

is known to be distributed as

$$p(y) = 2 \quad e^{-y} - e^{-2y} \qquad y > 0$$

$$= 0 \qquad y < 0$$

and that the frequency spectrum of the input is known to be much wider than that of the filter.

The z-transform transfer function corresponding to the given G(s) is

$$G(z) = \frac{K\Delta t \quad e^{-\Delta t} z}{(z - e^{-t})^2}$$

which for $\Delta t = 1$

$$= \frac{ke^{-1} \quad z^{-1}}{1 - 2e^{-1}z^{-1} + e^{-2}z^{-2}} .$$

If, for convenience ke^{-1} is set equal to unity, this corresponds to the difference equation:

$$x_n = 0.736x_{n-1} - 0.135x_{n-z} + z_{n-1}$$

The subscript n-1 on the z term is irrelevant here, however, since the distribution of z_{n-1} is the distribution of the complete input process. This equation implies that

$$\alpha_1 = 0.736$$

$$\alpha_2 = -0.135$$

and, therefore, that,

$$\phi = \frac{\alpha_1}{1-\alpha_2} = 0.85$$

$$\phi_2 = \alpha_1\phi_1-\alpha_2 = 0.49.$$

Noting that the transform of p(s) is

$$P_x(s) = \frac{2}{(s+1)(s+2)}$$

Equation 52 states that the characteristic function of the input distribution is given by;

$$P_z(s) = \frac{P_x(s)}{P_x(0.7365)} \quad \frac{P_x(0.625s)}{P_x(0.49s)}$$

$$= \frac{(0.736s+1\ )}{(s+1)} \frac{(0.736s+2)}{(s+2)} \quad \frac{(0.49s+1)}{(0.625s+1)} \quad \frac{(0.49s+2)}{(0.625s+2)}$$

$$= 0.332 \quad (\frac{s+1.36}{s+1})(\frac{s+2.72}{s+2})(\frac{s+2.04}{s+1.6})(\frac{s+4.08}{s+3.2})$$

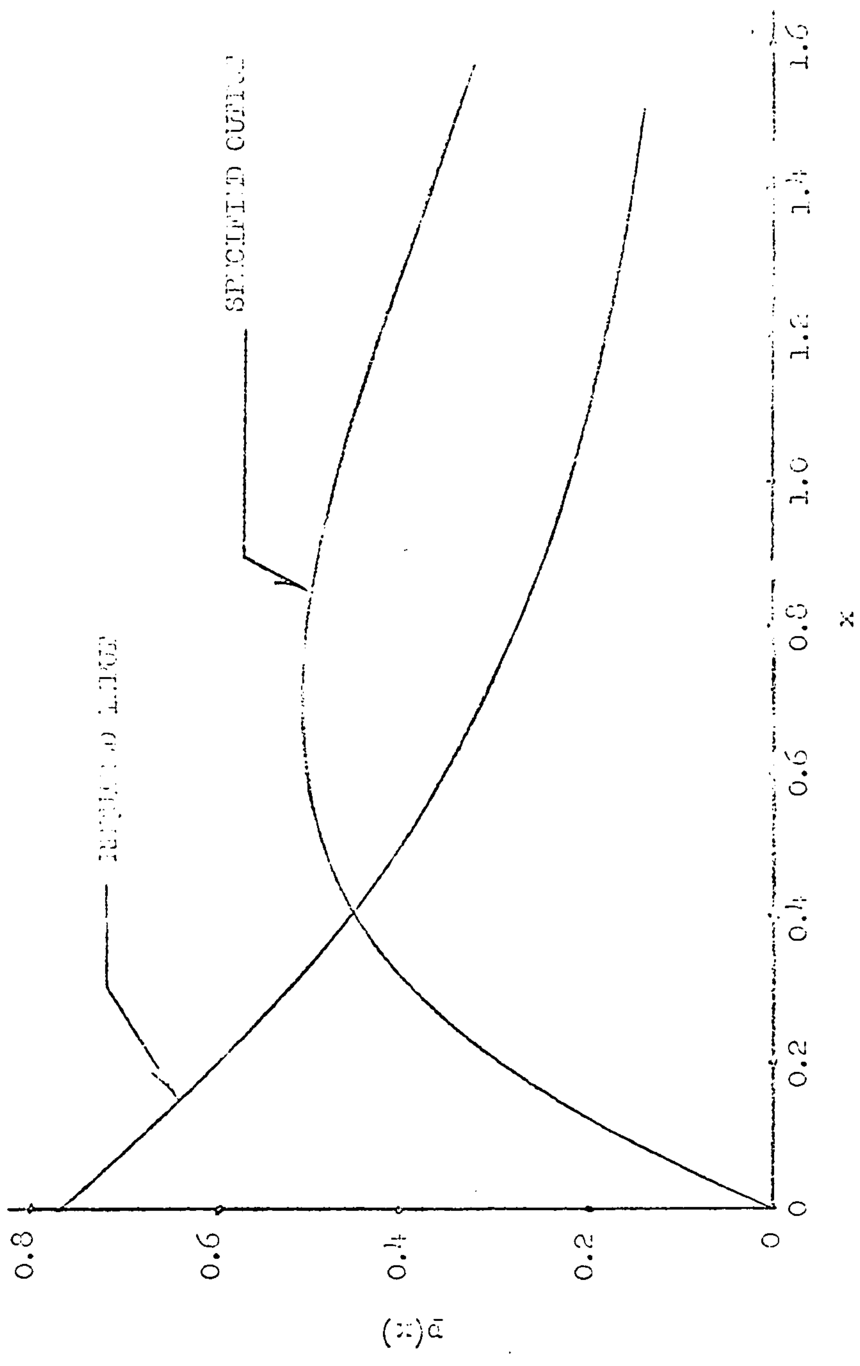
$$= 0.332 \left[1 + \frac{1.49}{s+1} - \frac{0.08}{s+2} + \frac{0.76}{s+1.6} + \frac{0.2}{s+3.2}\right]$$

The inverse of this is

$$0.332 \left[\delta(x) + 1.49e^{-x}-0.08e^{-2x}+0.76e^{-1.6x}+ 0.21e^{-3.2x}\right]$$

and represents the distribution that the input process must be. This function, as well as the given output distribution, is shown in Figure 8.

Figure 8. Shows the predicted input distribution and known output distribution for a filter with transfer function $G(s) = \frac{2.718}{(s+1)^2}$ as calculated in Example 7


0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
1.0
x

VI. ERRORS

Since the models treated are only approximate representations of continuous input processes, and the implied periodic sampling of the output (Equation 28) introduces further error, it would be desirable to determine the error of representation of the output distribution. However, because the criteria for error might vary widely from problem to problem and any quantitative analysis of closeness of fit would be legitimate only for each particular distribution, no systematic examination of representation error was made. Rather, a specific problem is solved for three different choices of Δt, to provide some feeling for the dependence of the shape of the output distribution on the selected input pulse width.

Assume again that the input distribution is

$$p(x) = e^{-x} \qquad x > 0$$

$$= 0 \qquad x < 0$$

and that the impulse response of the filter is

$$y(t) = 1/3\ e^{-t/3}$$

also assume the autocorrelation function is

$$\phi(\tau) = e^{-|\tau|}$$

as in Example 5. Then for the three different approximations defined by

a). $\Delta t = 2,\ \alpha = 0$

b). $\Delta t = 1.5,\ \alpha = 0.223$

c). $\Delta t = 1.0,\ \alpha = 0.368$

and shown in Figure 8, the output distributions are shown in Figure 9. It is seen that the change from $\Delta t = 1.5$ to $\Delta t = 1.0$ did not have a great effect on the distribution shape.

Figure 9. Shows three approximations to the autocorrelation function $\phi(\tau) = e^{-|\tau|}$ for a continuous function by different choices of Δt and α. The first case, $\Delta t = 2$, $\alpha = 0$, represents the model process discribed in (6). The other two cases illustrate models described in this paper

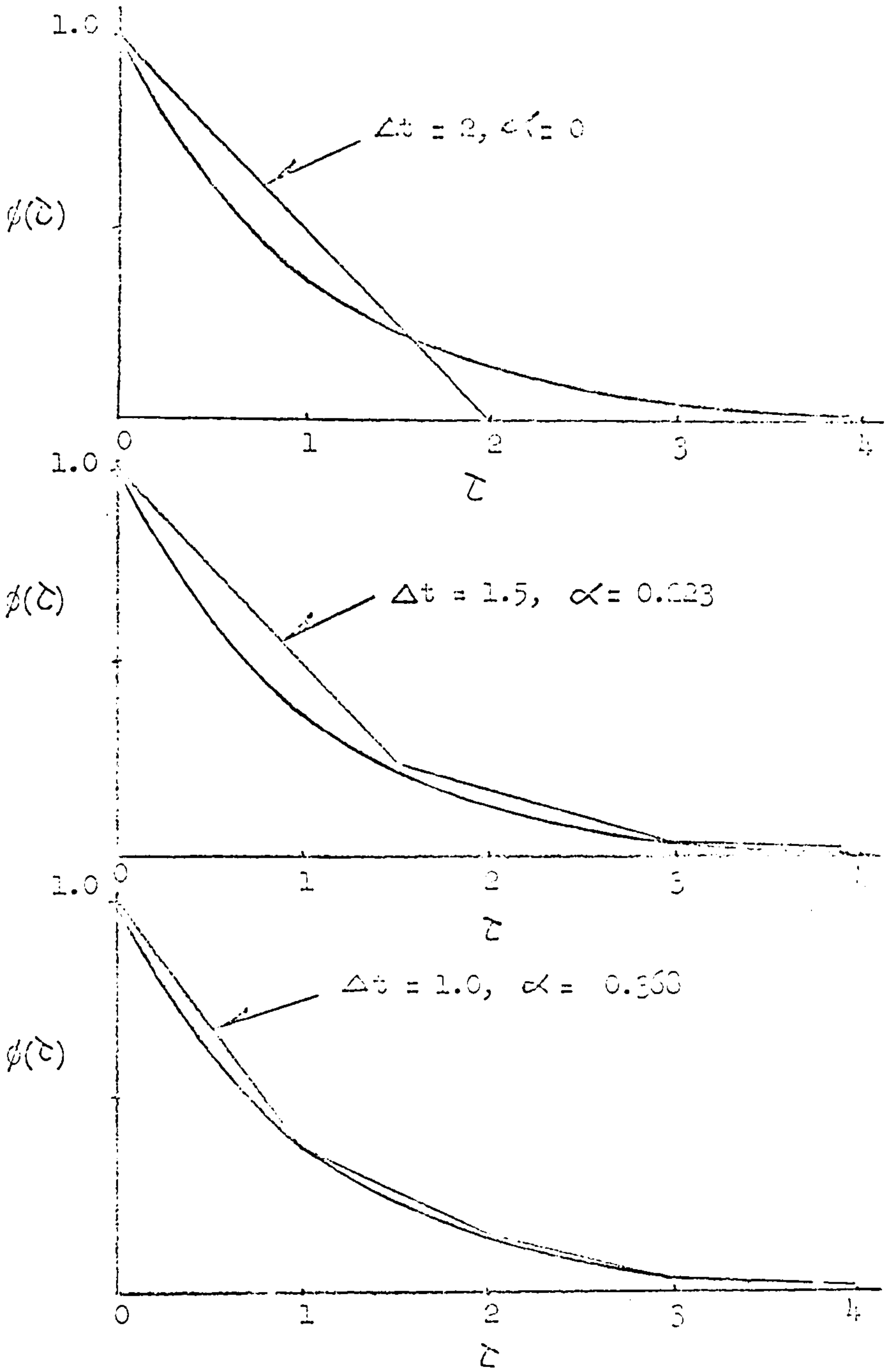
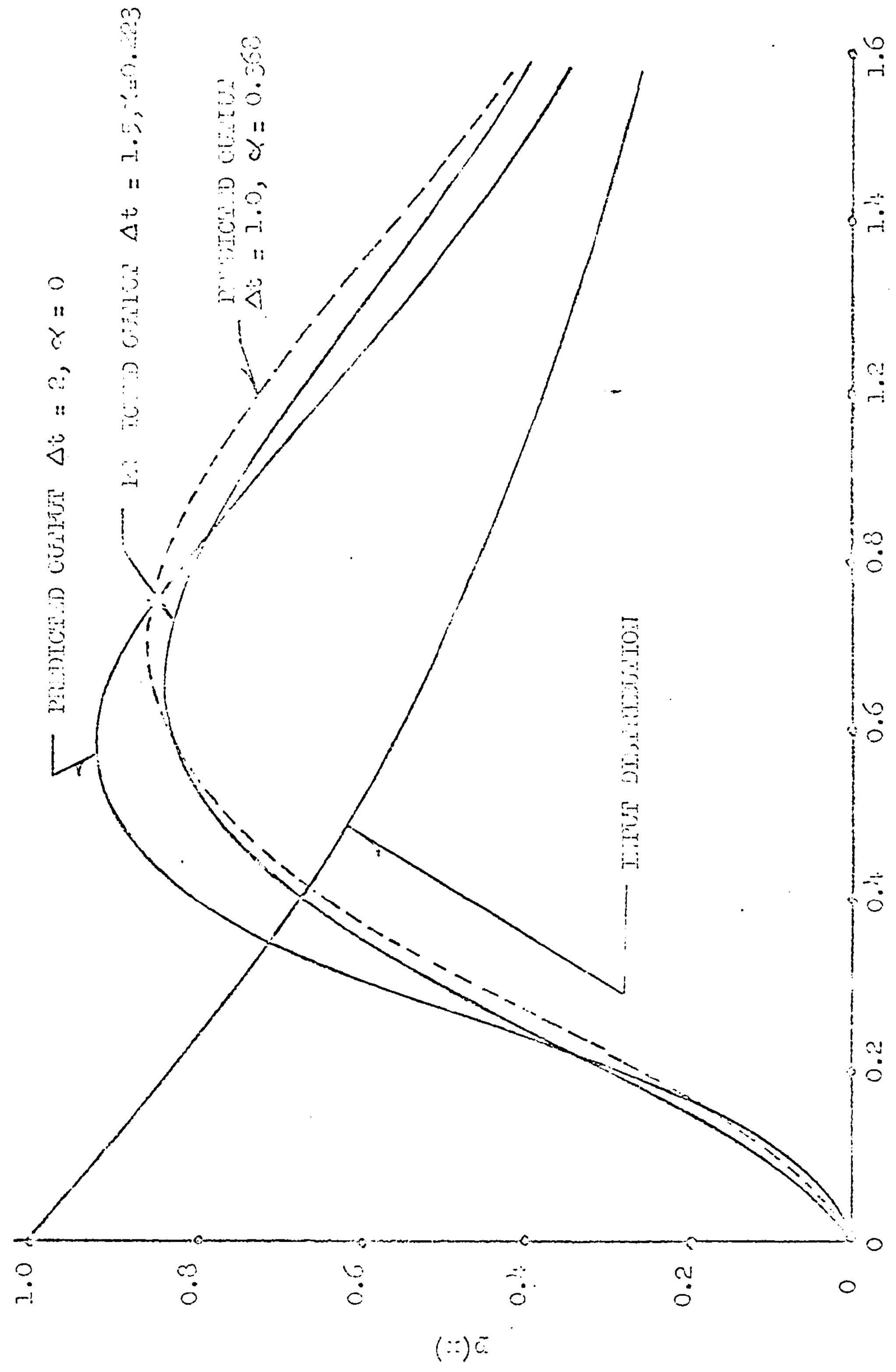

1.0
Δt = 2, α = 0
φ(τ)
0
1
2
3
4
τ
1.0
φ(τ)
Δt = 1.5, α = 0.223
0
1
2
3
τ
1.0
Δt = 1.0, α = 0.368
φ(τ)
0
1
2
3
4
τ

Figure 10. Shows the predicted output distribution of a first-order filter by three different models of the same continuous process. The autocorrelation functions for these models as compared to that for the continuous process are shown in Figure 9


PREDICTED OUTPUT Δt = 2, α = 0
Δt = 1.0, α = 0.368
INPUT DISTRIBUTION
1.0
0.8
0.6
0.2
0
0.2
0.4
0.6
0.8
1.2
1.6
x

VII. LOOSE ENDS

It might be useful to also note some of the problem areas where solutions were attempted but with less or no success.

A. Axis-Crossings.

The axis-crossing problem for other than first-order Markov processes was not solved because no way was found to transform the integral

$$\iiint_{x_1 x_2 x_3} P\left\{x_1, x_2, x_3\right\} dx_1 dx_2 dx_3$$

when

$$P\left\{x_3 \middle| x_2, x_1\right\} \neq P\left\{x_3 \middle| x_2\right\} = P_1(x_3 - k_1 x_2).$$

If an analytic expression could be found for this integral, higher-order processes could be handled also. Of particular interest would be the axis-crossing intervals of band-limited noise with or without an additive, known wave form.

B. Higher-Order Models.

Treatment of a model process where

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \alpha_3 x_{n-3} + z_n$$

was also attempted with some success but considerably more cumbersome computations. This problem was resolved to the point of determing the distribution of the generating variable z to be

$$P_{(z)} = \frac{P_x(w)}{P_x(\phi_3 w)} \; \frac{P_x(m_1 \phi_1 w) \; P_x(\alpha_1 l_1 w) \; P_x(\alpha_1 \phi_2 w)}{P_x(m_1 w) \quad P_x(\alpha_1 l_1 \phi_1 w) \; P_x(\alpha_1 w)}$$

where m_1 and l_1 are constants. No way was found to determine these constants however, with only the autocorrelation function for the continuous process given. It is felt that this is philosophically consistent with requiring both

$$W_2 = \iint_{x_1 x_2} x_1 x_2 p_2(x_1, x_2)\ dx_2 dx_1$$

and

$$W_3 = \iiint_{x_1 x_2 x_3} x_1 x_2 x_3 p_3(x_1, x_2, x_3)\ dx_1 dx_2 dx_3$$

to define a third-order process. Thus, if W_3 were known for a third-order process, it could also be handled with this technique.

C. More Complex Models.

Section IV discusses some of the limitations imposed on the form of the probability density functions allowed by this model. It is not difficult to conceive of other, more complicated models which would probably reduce these restrictions, and an attempt to formulate some of them was made. However, in each case, the treatment of the model became insufferably unwielding and complex. Perhaps other models exist which are little or no more difficult to handle, but they were not found.

VIII. SUMMARY AND CONCLUSIONS

A. Specific Results.

The rectangular-pulse-model process was found to be an effective vehicle for obtaining approximate solutions to many problems. Some specific examples treated were;

1). Determination of the distribution of the output of any linear filter with arbitrary precision, when the input process is continuous and has a first- or second-order autocorrelation function.

2). Determination of the first conditional probability density function, or diffusion equation, for a continuous process specified by an autocorrelation function and probability density function.

3). Discription of the axis-crossing durations for any continuous Markov process.

4). Determination of the input distribution of a first- or second-order filter, given the output distribution, if the input is known to be essentially white.

B. Indicated Areas of Extension.

In addition to the specific results mentioned above, the techniques outlined seem applicable to the many other related problems.
Some example are;

1). Determing the distribution of the axis-crossing intervals associated with a known wave-form and additive Markov noise.

2). The linear filter response to a discrete-level-input process.

3). The distribution of a filter output for continuous inputs with autocorrelation functions of order greater than two.

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